Solution 10

1. Show that the bounded sequence of sequences $\{\mathbf{e}_n\}$ where $\mathbf{e}_n = (0, \cdots, 0, 1, 0, \cdots,)$ is the sequence with 1 at the n-th place and equal to 0 elsewhere has no convergent subsequences in the space l^2. Recall that l^2 is the space consisting of all sequences $\mathbf{a} = \{a_n\}$ satisfying $\|\mathbf{a}\|_2 = (\sum_n a_n^2)^{1/2} < \infty$

   Solution. Suppose on the contrary this sequence has a limit $\mathbf{a} = \{a_n\}$. (I have used bold letters to denote sequences.) Then $\lim_{n\to\infty} \|\mathbf{e}_n - \mathbf{a}\| = 0$. From the definition of the l^2-norm it means every component of $\mathbf{e}_n - \mathbf{a}$ tends to zero. Since the k-component of $\mathbf{e}_n$ becomes zero when $n > k$, the sequence $\mathbf{a}$ must be the zero sequence. Therefore, in case the sequence formed by $\mathbf{e}_n$'s has a convergent subsequence, it also converges to the zero sequence in the l^2-norm. But this is impossible since $\lim_{n\to\infty} \|\mathbf{e}_{n_k} - \mathbf{0}\| = 1$.

2. Consider $\{f_n\}, f_n(x) = x^{1/n}$, as a subset $\mathcal{F}$ in $C[0, 1]$. Show that it is closed, bounded, but has no convergent subsequence in $C[0, 1]$.

   Solution. It means $\mathcal{F}$ is not precompact. $\mathcal{F}$ is bounded as $\|f_n\|_\infty \le 1$ for all $f \in \mathcal{F}$. Next, we claim that it has no convergent subsequence. Suppose on the contrary there is one subsequence $\{f_{n_j}\}$ converges to some $g \in C[0, 1]$. Then, for each x, one must have $\lim_{j\to\infty} f_{n_j}(x) = g(x)$. However, it is clear that the pointwise limit of f_n is the function $f(x) = 1, x \in (0, 1]$ and equals 0 at $x = 0$. So g must coincide with f, but this is impossible as g is continuous on $[0, 1]$ but f is discontinuous at $x = 0$.

   We still need to check that $\mathcal{F}$ is closed. Let $\{h_n\}$ be a sequence in $\mathcal{F}$ converging to some $h \in C[0, 1]$. Consider two cases. First, this sequence contains infinitely many distinct functions. Then we can extract a subsequence from it which is also a subsequence of $\{f_n\}$. As above we see that this is impossible because h is continuous but f is not. Second, $\{h_n\}$ contains only finitely many functions. Then one function, say, f_{n_0}, appears infinitely many times. We can take a subsequence $\{h_{n_j}\}$ consisting of the single f_{n_0}. It must be true that $h = f_{n_0} \in \mathcal{F}$. We conclude that $\mathcal{F}$ is a closed set.

3. Prove that $\{\cos nx\}_{n=1}^\infty$ does not have any convergent subsequence in $C[0, 1]$.

   Solution. By Arzela Theorem it suffices to show that this sequence has no subsequence that is equicontinuous. Suppose on the contrary, given $\varepsilon > 0$, there exists some $\delta > 0$ such that
   $$|\cos n_k x - \cos n_k y| < \varepsilon, \quad \forall k \ge 1,\ x, y,\ |x - y| < \delta.$$
   Now, take $\varepsilon = 1$ so δ is fixed. Take $x = 0$ and $y = \pi/n$. When n is large $|0 - \pi/n| < \delta$, one should have $|\cos n0 - \cos n\pi/n| < \varepsilon = 1$. But actually we have $|\cos n0 - \cos n\pi/n| = 2$, contradiction holds.

4. Show that any finite set in $C(\overline{G})$ is bounded and equicontinuous.

   Solution. Recall that any continuous function in $\overline{G}$ is uniformly continuous. (The proof is similar to the special case $C[a, b]$.) Now, let the finite set be $\{f_1, \cdots, f_N\}$. Since each f_k is uniformly continuous, for $\varepsilon > 0$, there is some δ_k such that $|f_k(x) - f_k(y)| < \varepsilon$ for all $x, y,\ |x - y| < \delta_k$. If we take $\delta = \min\{\delta_1, \cdots, \delta_N\}$. Then $|f_k(x) - f_k(y)| < \varepsilon$ for $x, y,\ |x - y| < \delta$ and all k. On the other hand, it is clearly bounded by the maximum of $\|f_1\|_\infty, \cdots, \|f_N\|_\infty$.

5. Let E be a bounded, convex set in $\mathbb{R}^n$. Show that a family of equicontinuous functions is bounded in E if it is bounded at a single point, that is, if there are $x_0 \in E$ and constant M such that $|f(x_0)| \leq M$ for all f in this family.

   Solution. By equicontinuity, for $\varepsilon = 1$, there is some δ_0 such that $|f(x) - f(y)| \leq 1$ whenever $|x - y| \leq \delta_0$. Let $B_R(x_0)$ a ball containing E. Then $|x - x_0| \leq R$ for all $x \in E$. We can find $x_0, \cdots, x_n = x$ where $n\delta_0 \leq R \leq (n+1)\delta_0$ so that $|x_{n+1} - x_n| \leq \delta_0$. It follows that

   $$|f(x) - f(x_0)| \leq \sum_{j=0}^{n-1} |f(x_{j+1} - f(x_j)| \leq n \leq \frac{R}{\delta_0}.$$

   Therefore,

   $$|f(x)| \leq |f(x_0)| + n + 1 \leq M + \frac{R}{\delta_0} \quad \forall x \in E, \ \forall f \in \mathcal{F}.$$

6. Let $\{f_n\}$ be a sequence of bounded functions in $[0,1]$ and let F_n be

   $$F_n(x) = \int_0^x f_n(t)dt.$$

   (a) Show that the sequence $\{F_n\}$ has a convergent subsequence provided there is some M such that $\|f_n\|_\infty \leq M$, for all n.

   (b) Show that the conclusion in (a) holds when boundedness is replaced by the weaker condition: There is some K such that

   $$\int_0^1 |f_n|^2 \leq K, \quad \forall n.$$

   Solution.

   (a) Since $|F_n| \leq \int_0^x |f_n(t)|dt \leq M$, and $|F_n(x) - F_n(y)| \leq \int_y^x |f_n(t)|dt \leq |x-y|M$, $\{F_n\}$ is uniformly bounded and equicontinuous. Then it follows from Arzela-Ascoli theorem that $\{F_n\}$ is sequentially compact.

   (b) It follows from the Cauchy-Schwarz inequality that

   $$|F_n(x) - F_n(y)| \leq \int_y^x |f_n(t)|dt \leq \left(\int_y^x 1^2 dt \right)^{1/2} \left(\int_y^x |f_n(t)|^2 dt \right)^{1/2} \leq \sqrt{K}\sqrt{|x-y|}.$$

   Similarly one can show that $\{F_n\}$ is uniformly bounded. Then apply Arzela-Ascoli theorem.

7. Prove that the set consisting of all functions G of the form

   $$G(x) = \sin^2 x + \int_0^x \frac{g(y)}{1 + g^2(y)} \, dy \ ,$$

   where $g \in C[0,1]$ is precompact in $C[0,1]$.

   Solution. Straightforward to check $\|G\|_{L^\infty} \leq 2$ and $\|G'\|_{L^\infty} \leq 3$. By Ascoli's Theorem this set is precompact.

8. Let $K \in C([a,b] \times [a,b])$ and define the operator T by

   $$(Tf)(x) = \int_a^b K(x,y)f(y)dy.$$

(a) Show that T maps $C[a,b]$ to itself.

(b) Show that whenever $\{f_n\}$ is a bounded sequence in $C[a,b]$, $\{Tf_n\}$ contains a convergent subsequence.

Solution.

(a) Since $K \in C([a,b] \times [a,b])$, given $\varepsilon > 0$, there exists $\delta > 0$ such that $|K(x,y) - K(x',y)| < \varepsilon$, whenever $|x - x'| < \delta$. Then for $x, x' \in [a,b], |x - x'| < \delta$, one has

$$|(Tf)(x) - (Tf)(x')| \le \int_a^b |K(x,y) - K(x',y)||f(y)|dy \le |a-b|\|f\|_\infty \varepsilon.$$

Hence $Tf \in C[a,b]$.

(b) Suppose $\sup_n \|f_n\|_\infty \le M < \infty$. It follows from the proof of (a) that δ can be taken independent of n. Hence $\{f_n\}$ is equicontinuous. Furthermore, since $|(Tf_n)(x)| \le \int_a^b |K(x,y)||f_n(y)|dy \le M(b-a)\|K\|_\infty$, $\{f_n\}$ is uniformly bounded. Then it follows from Arzela-Ascoli theorem that $\{Tf_n\}$ contains a convergent subsequence.

9. Let f be a bounded, uniformly continuous function on $\mathbb{R}$. Let $f_a(x) = f(x+a)$. Show that for each $l > 0$, there exists a sequence $\{a_n\}$, $a_n \to \infty$, such that $\{f_{a_n}\}$ converges uniformly on $[0,l]$.

Solution. Let $\{a_n\}$ be a sequence with $a_n \to \infty$. Since f is bounded and uniformly continuous on $\mathbb{R}$, it follows that $\{f_{a_n}\}$ is uniformly bounded and equivcontinuous on $[0,l]$. Apply Ascoli-Arezela theorem to obtain a subsequence converging uniformly on $[0,l]$.

Note. The lesson is, if you keep watching dramas in TVB every evening, soon you find some new one resembling an old one.

10. Optional. Let $\{h_n\}$ be a sequence of analytic functions in the unit disc satisfying $|h_n(z)| \le M, \ \forall z, |z| < 1$. Show that there exist an analytic function h in the unit disc and a subsequence $\{h_{n_j}\}$ which converges to h uniformly on each smaller disc $\{z : |z| \le r\}$, $r \in (0,1)$. Suggestion: Use a suitable Cauchy integral formula.

Solution. Let $D = \{|z| < 1\}$ be the unit disc. Let $r_n \uparrow 1$ be strictly increasing and $D_n = \{|z| < r_n\}$. For each $n \in \mathbb{N}$, since h_j is analytic in D, it follows from the Cauchy integral formula that

$$h_j(z) = \frac{1}{2\pi i} \int_{|\zeta| = r_{n+1}} \frac{h_j(\zeta)}{\zeta - z} d\zeta, \qquad \text{for } |z| < r_n.$$

Hence

$$|h_j'(z)| = |\frac{1}{2\pi i} \int_{|\zeta| = r_{n+1}} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta| \le \frac{M}{2\pi |r_{n+1} - r_n|^2}, \qquad \text{for } |z| < r_n.$$

Since $|h_j'(z)|$ is uniformly bounded on D_n, it follows that $\{h_j\}$ is equicontinuous on each D_n. Applying Ascoli-Arezela theorem to $\{h_j\}$ on each D_n step by step and then taking a Cantor's diagonal sequence, one obtains a $\{h_{n_j}\}$ which converges to h uniformly on each smaller disc $\{z : |z| \le r\}$, $r \in (0,1)$. It follows from uniform convergence that

$$h(z) = \frac{1}{2\pi i} \int_{|\zeta| = r_{n+1}} \frac{h(\zeta)}{\zeta - z} d\zeta, \qquad \text{for } |z| < r_n.$$

Hence h is analytic in D.